1 Overview of Linear Transformations & Projections

In this paper, we explore the underlying math of perspective projections. In particular, how 3-dimensional objects are captured on a 2-dimensional plane. This entails computing lines through a fixed point called the *the center of projection*, and understanding where these lines meet a certain fixed 2-dimensional plane called the *viewing plane*.

1.1 Recap of Linear Transformations and Orthogonal Projections

Linear transformations are functions that take in a vector and output another transformed vector via matrix multiplication. For example, the function $T(x, y) = (2x + 3y, -x + 2y)$ is a linear transformation that can be represented by multiplying input vectors in $\mathbb{R}^2$ with the matrix

$$\begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}.$$

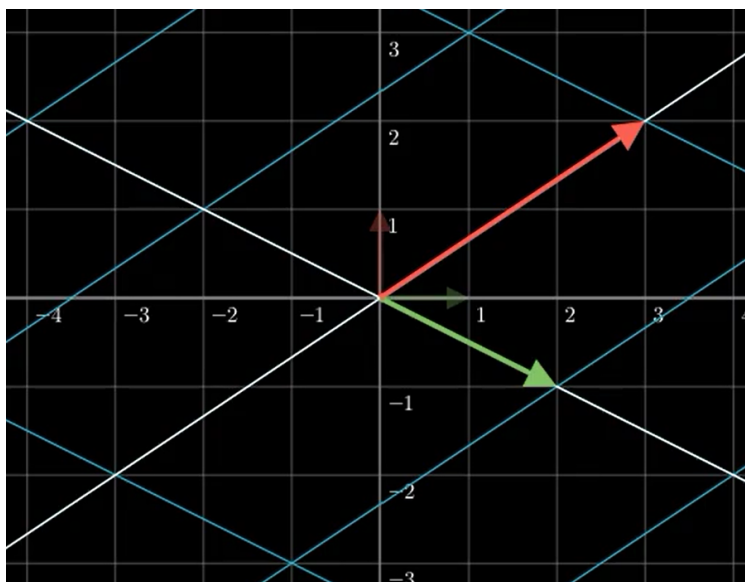Geometrically, this function reshapes our standard $x - y$ plane like this:

Figure 1: $T(x, y) = (2x + 3y, -x + 2y)$. Notice how $(1, 0)$ goes to $(2, -1)$ and $(0, 1)$ goes to $(3, 2)$.

A defining feature of linear transformations is that they abide by the rules of linearity. A key consequence of this is that our grid lines always stay evenly spaced and parallel. That is, linear transformations map linear combinations of input to linear combinations of output. Natural examples of linear transformations are rotations about an axis, reflections about a line, and scaling. For example, the matrix

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

represents the linear transformation that takes in a vector (x, y) and spits out the input vector rotated clockwise by $90°$.

Orthogonal projections are another interesting class of linear transformations. For example, orthogonal projections onto a 2-dimensional plane in $\mathbb{R}^3$ collapse lines perpendicular to the plane of projection to a single point, while points on the 2-dimensional plane stay fixed. One feature of projection transformations

is that they are examples of idempotent transformations, namely applying the projection more than once returns the same result as applying the projection once. Let's look at the following projection

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

which takes in a vector (x, y, z) and projects onto the to the point $(x, y, 0)$. Lines perpendicular to the plane of projection collapse to single points. Lines that are not perpendicular to the xy-plane map to lines in $\mathbb{R}^3$. More importantly, *parallel* lines in $\mathbb{R}^3$ map to other *parallel* lines in $\mathbb{R}^3$ contained in $\mathbb{R}^3$, that is, unless they are perpendicular to the plane of projection. In the following graphic, we see what happens to a sphere in $\mathbb{R}^3$ under such projection:
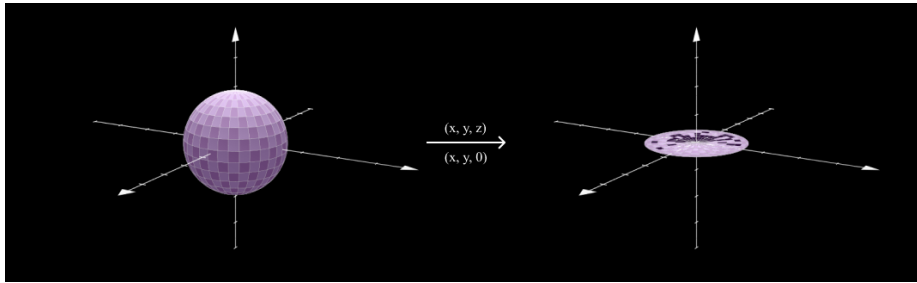

Figure 2: An example of a sphere under orthogonal projection.

Like orthogonal projections, Perspective Projections move input vectors onto a projected space, and they are also idempotent because points already in the projected space remain unchanged. But unlike orthogonal projections, perspective projections are non-linear and do not map parallel lines to parallel lines. The key difference is that perspective projections take into account a fixed point, or perspective.

1.2 Introduction to Perspective Projections

Imagine we take a picture of the physical space we experience before us. The picture captures objects existing in our 3-dimensional space, and it projects them into 2-dimensional space. Looking at the picture, objects farther from the camera appear smaller while objects closer to the camera appear larger. This appearance is natural, but we wish to understand the math going on under the hood.

Perspective projections have a fitting name. They are *projections*, because they are idempotent — points on the viewing plane are fixed under perspective projection. And they are *perspective*, because they entail a fixed point from which the projection occurs. In the aforementioned example, the camera lens would be the fixed point, or 'center of projection', and the 2-dimensional picture would represent the new dimension $\mathbb{R}^2$, or 'viewing plane'.

     Attributes of Perspective Projections:
- ❖ Center of Projection - the perspective from which projection occurs
- ❖ Viewing Plane - the space projected onto

Before getting into the weeds of the math, let's think about our objective: to figure out where a point in the domain is sent to on the viewing plane after perspective projection. In order to do this, we connect lines between the point that we want to project, our center of projection, and our viewing plane. By sliding the point we wish to the project — over the line — to the viewing plane, we get our projected point on the viewing plane. Regarding the computation in this way, it becomes apparent that if we were to project a point already existing on the the viewing plane, it would remain unchanged. The following figure nicely

illustrates a perspective projection and its relevant lines between the center of projection and the viewing screen.
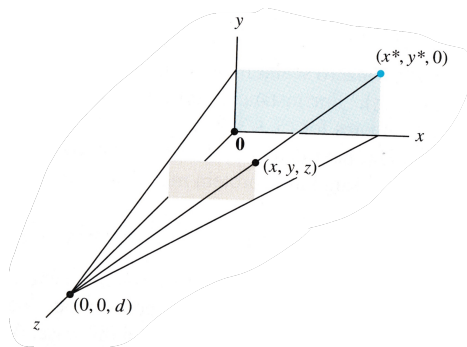

Figure 3: Projecting (x, y, z) from center of projection $(0, 0, d)$ to viewing plane $z = 0$.

Suppose we want to know where the point (x, y, z) goes under the projection depicted above. In order to map this, we would first need to define the line joining the point (x, y, z) and the center of projection $(0, 0, d)$, then slide along that line until it meets the viewing plane $z = 0$. Let's take a more thorough look into the math.

2 Math of Perspective Projections

Computing perspective projections is first contingent upon defining the center of projection, the viewing plane, and the lines that lie in between them. The following section provides general information for describing lines and planes in the language of linear algebra. Then, the succeeding section details the perspective projection recipe!

2.1 Describing Lines and Planes in $\mathbb{R}^3$

A 2-dimensional plane in $\mathbb{R}^3$ through the origin can either be implicitly described as the solutions to a single equation, such as $x + y = 0$ in three variables x, y, z, or described explicitly (i.e., parametrically) as the collection of linear combinations of the two vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ that lie on this plane.

In this paper, we will focus on describing lines explicitly. Lines through the origin in $\mathbb{R}^3$ can be described as multiples of a single nonzero vector in $\mathbb{R}^3$. For example, the points

$$\left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \text{ any number} \right\}$$

describe a line. The explicit description of a line parallel to the line above but passing through the point $(4, -4, 5)$ is

$$\left\{ \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \text{ any number} \right\}.$$

Similarly, one may also explicitly describe the line joining two points in $\mathbb{R}^3$. For example, the line connecting

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two points $P : \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix}$ and $Q : \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix}$ is given by

$$\left\{ (1-t) \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 5 \\ -3 \\ 5 \end{pmatrix} : t \text{ any number} \right\} = \left\{ \begin{pmatrix} 4 \\ -4 \\ 5 \end{pmatrix} + t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} : t \text{ any number} \right\}.$$

The point P corresponds to the value of the parameter $t = 0$ and the point Q corresponds to the value of the parameter $t = 1$. With these technicals reviewed, we are ready to solve for a perspective projection.

2.1.1 Parallel Lines Under Linear Transformations

There is a key difference in the mapping of parallel lines under linear transformations, as opposed to under perspective projections. Linear transformations satisfy the following equation for any two vectors a, v and any scalar t:

$$T(a + tv) = T(a) + tT(v).$$

By linearity, linear transformations map linear combinations of input to linear combinations of output. In general, the line through the point a in the direction of the vector v, under a linear transformation T, becomes the line through the point $T(a)$ in the direction of vector $T(v)$. Succinctly, lines go to lines if v is not in the null space of T (i.e., when $T(v) \neq 0$), or lines go to a single point if the direction v is in the null space of T (when $T(v) = 0$).

Thus, parallel lines map to parallel lines (or points) under linear transformations. Two lines are parallel if they correspond to the same direction vector v but a different a. After applying the linear transformation, they become lines in the same direction $T(v)$.

Let's visualize this by considering the orthogonal projection onto the xy-plane, and seeing what happens to parallel lines after such projection. In general, we'll shorthand an orthogonal projection as follows:

$$\text{OP} \left(\begin{pmatrix} a \\ b \\ c \end{pmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ 0 \end{pmatrix}.$$

Now take the 3 parallel lines:

$$t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ and } t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Apply orthogonal projection, and we see linearity plays a key role:

$$\text{OP} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$$\text{OP} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \text{OP} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) + \text{OP} \left(\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix};$$

$$\text{OP} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = \text{OP} \left(t \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right) + \text{OP} \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) = t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

These parallel lines with direction vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ mapped to parallel lines with direction vector $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ under orthogonal projection to the xy-plane. Note that this is the case where the direction vector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ corresponding to the input lines is not in the null space of the orthogonal projection transformation, so the images of these lines are once again lines (and not points).

If we instead consider three parallel lines with direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ parametrized by t as before:

$$t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \text{ and } t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

We get:

$$\text{OP}\left(t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

$$\text{OP}\left(t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \text{OP}\left(t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) + \text{OP}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix};$$

$$\text{OP}\left(t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \text{OP}\left(t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}\right) + \text{OP}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Note that there is no longer a parameter t in the output vectors — each of these parallel lines collapses to a single point under the orthogonal projection transformation because the direction vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is in the null space of orthogonal projection onto the xy-plane.

We will see that the same is not true for perspective projections. Although lines still map to lines or points under perspective projections (even this is not immediately clear from the algebraic description!), parallel lines no longer map to parallel lines under perspective projection transformations.

2.2   Perspective Projection Transformation

Suppose we wish to compute *the perspective projection that moves points in $\mathbb{R}^3$ to the plane $x + y = 0$, from the point $(d, d, 0)$*. In this case, the center of projection is $(d, d, 0)$, and the viewing plane is the plane $x + y = 0$ in $\mathbb{R}^3$, which is the implicit definition of the 2-dimensional plane in $\mathbb{R}^3$ that contains the line $y = -x$ in $\mathbb{R}^2$ and the vector $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Explicitly, this plane is the span of the two vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.

Notice too that we choose a variable center of projection. This is practical because, in real application, all perspectives are different. The line that passes through the points $(d, d, 0)$ and (x, y, z) is defined as

$$(1-t)\begin{pmatrix} d \\ d \\ 0 \end{pmatrix} + t\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} d - td + tx \\ d - td + ty \\ tz \end{pmatrix}.$$

The line is parameterized with a scalar t. In general, this parameter t could be thought of as an iterator that slides along all the points of the line. Relative to the perspective projection, the value of t corresponds to the unique point where this line — from the center of projection to any point (x, y, z) — intersects the viewing plane. We solve for this value t using algebra — by plugging the coordinates of the point on the line into the equation $y = -x$ (our viewing plane!):

$$d - td + ty = -d + td - tx.$$
$$t(x - 2d + y) = -2d.$$
$$t = \frac{-2d}{x - 2d + y}.$$

Then to find the point in the line corresponding to this value of t, plug t back into the parameterized equation of the line:

$$\begin{pmatrix} d + \frac{2d^2}{x-2d+y} - \frac{2dx}{x-2d+y} \\ d + \frac{2d^2}{x-2d+y} - \frac{2dy}{x-2d+y} \\ \frac{-2dz}{x-2d+y} \end{pmatrix} = \begin{pmatrix} \frac{d(y-x)}{x-2d+y} \\ \frac{d(x-y)}{x-2d+y} \\ \frac{-2dz}{x-2d+y} \end{pmatrix}. \tag{Eq. 1}$$

That's it, the vector above represents the output of the function that projects the input vector (x, y, z) onto the plane $x + y = 0$ from the point $(d, d, 0)$. We'll call it *Eq. 1* as a reference for later.

2.2.1 Nonlinearity of Projections and a Linear Hack: Homogeneous Coordinates

Perspective projections do not abide by the rules of linearity. Instead there is a hack to represent a nonlinear perspective projection transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ as a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$. This is called employing homogeneous coordinates. The key idea that a single point (x, y, z) in $\mathbb{R}^3$ gets represented by *infinitely* many points in $\mathbb{R}^4$ of the form (xh, yh, zh, h) for any nonzero h — it is this extra flexibility with multiple representatives that lets us write down a linear map. So for example, the point $(2, 3, 4)$ in $\mathbb{R}^3$ is represented by $(2, 3, 4, 1), (4, 6, 8, 2), (1, 3/2, 2, 1/2)$, etc in $\mathbb{R}^4$. For the hack, we represent each point $(x, y, z) \in \mathbb{R}^3$ as $(x, y, z, 1) \in \mathbb{R}^4$. Now by using homogeneous coordinates to scale out the denominator, we apply perspective projection on $(x, y, z, 1) \in \mathbb{R}^4$ to get $(d(y-x), d(x-y), -2dz, x+y-2d) \in \mathbb{R}^4$. When we convert back to regular coordinates in $\mathbb{R}^3$ from homogeneous coordinates in $\mathbb{R}^4$, we get the output of the perspective projection transformation.

Hence, the extra direction provides a 'constant that we can scale and add to the other coordinates, as needed, via matrix multiplication' (Crovella). The perspective projection transformation

$$\begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} d(y-x) \\ d(x-y) \\ -2dz \\ x+y-2d \end{pmatrix}$$

is represented by the following matrix

$$\begin{bmatrix} -d & d & 0 & 0 \\ d & -d & 0 & 0 \\ 0 & 0 & -2d & 0 \\ 1 & 1 & 0 & -2d \end{bmatrix}.$$

Besides making perspective projections linear, this linearity sheds light on a feature of perspective projections that is not immediately apparent from the formulaic or geometric description. Because a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^4$ can be completely reconstructed from knowing where four linearly independent vectors in $\mathbb{R}^4$ go, if we know the images of four sufficiently general points in $\mathbb{R}^3$ under perspective projection (i.e., when the corresponding points in $\mathbb{R}^4$ using homogeneous coordinates are linearly independent vectors in $\mathbb{R}^4$), then we can reconstruct and say where every point in $\mathbb{R}^3$ goes.

3 Features of Perspective Projections

Now that we have a map, we can play around with some coordinates and see how they are affected by the projection and explain various features of perspective projections.

3.1 Perspective Projections are Idempotent

We can immediately show that perspective projections are idempotent. Suppose we define $d = 4$, making our center of projection $(4, 4, 0)$, and suppose we plugged in the point $(2, -2, 5)$. After applying the projection,

$(2, -2, 5)$ remains unchanged, because it is already on the viewing plane $y = -x$:

$$\begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix} \frac{4(-2-2)}{2-8-2} \\ \frac{4(2+2)}{2-8-2} \\ \frac{-2(4)(5)}{2-8-2} \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 5 \end{pmatrix}.$$

More generally, we see (as in the example above) that plugging in any point $(x, -x, z)$, which describes a general point on the plane $x + y = 0$ in $\mathbb{R}^3$, we get back the point $(x, -x, z)$, in exactly the same way:

$$\begin{pmatrix} x \\ -x \\ z \end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix} \frac{d(-x-x)}{x-2d-x} \\ \frac{d(x-(-x))}{x-2d-x} \\ \frac{-2dz}{x-2d-x} \end{pmatrix} = \begin{pmatrix} \frac{-2dx}{-2d} \\ \frac{2dx}{-2d} \\ \frac{-2dz}{-2d} \end{pmatrix} = \begin{pmatrix} x \\ -x \\ z \end{pmatrix}.$$

This shows that the perspective projection transformation is idempotent. This is also apparent geometrically: the point on the viewing plane and the line that connects the center of projection with the viewing plane is of course the same point.

3.2   Domain of Perspective Projections

It should be noted that the function (Eq. 1) is undefined when the denominator of the output vector $x - 2d + y$ becomes 0. The solution set to the denominator $x - 2d + y = 0$ describes a plane parallel to the viewing plane. In parametric form with parameters t and s, the solution set to $x - 2d + y = 0$ is $(d, d, 0) + t(1, 1, 0) + s(0, 0, 1)$. It is the span of the two vectors $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, translated to pass through the center of projection $(d, d, 0)$. This makes sense because the line joining the center of projection and another point on the plane parallel to the viewing plane, translated by the center of projection, would never intersect with the viewing plane.

3.3   Farther Objects Appear Smaller

Why do objects farther away from the center of projections appear smaller than those closer to the center of projection? To frame this question linearly, we can create a set of objects (consisting of lines) — or collection of points — at different distances from the center of projection, plug them into the projection function, and see how the objects transform. Now suppose we fixed our center of projection at $(-1, -1, 0)$, and projected the following set of parallel line segments onto $x + y = 0$:

$$t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}, \ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix}, \text{ and } t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 200 \\ 200 \\ 0 \end{pmatrix} \text{ for } -1 \leq t \leq 1.$$

1) Plug the line segment $\left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} : -1 \leq t \leq 1 \right\}$ into the projection function:

$$t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} -t + 2 \\ t + 2 \\ 0 \end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix} \frac{-1((t+2)-(-t+2))}{(-t+2)+2+t+2} \\ \frac{-1(-t+2-(t+2))}{(-t+2)+2+t+2} \\ \frac{-2(-1)(0)}{(-t+2)+2+t+2} \end{pmatrix} = \begin{pmatrix} \frac{-2t}{6} \\ \frac{2t}{6} \\ 0 \end{pmatrix} = t \begin{pmatrix} \frac{-2}{6} \\ \frac{2}{6} \\ 0 \end{pmatrix}. \tag{Eq. 2}$$

Note that the output consists of all multiples of a fixed vector, so the original line maps to a line through the origin under perspective projection transformation. The line segment input mapped to a line segment output (because t is a scalar parameter with a fixed range). That is, the line is represented by the series of

vectors when t is plugged in and $-1 \leq t \leq 1$. To make the comparison with the original line segment, we now compute the end points of this output line segment by plugging $t = -1$ and $t = 1$ into (Eq. 2):

$$\text{for } t = -1, \text{it gives } \begin{pmatrix} \frac{1}{3} \\ \frac{-1}{3} \\ 0 \end{pmatrix} \text{ and for } t = 1, \text{it gives } \begin{pmatrix} \frac{-1}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix}.$$

Originally, the line segment extended from $(3, 1, 0)$ to $(1, 3, 0)$, but after projection it spans from $(1/3, -1/3, 0)$ to $(-1/3, 1/3, 0)$. The object shrunk; the original length was $2\sqrt{2}$, but after projection the length is $(1/3)\sqrt{2}$. Now let's plug the other further away lines and see what happens to them.

2) Plug the line segment $\left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} : -1 \leq t \leq 1. \right\}$ in:

$$t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 20 \\ 20 \\ 0 \end{pmatrix} = \begin{pmatrix} -t + 20 \\ t + 20 \\ 0 \end{pmatrix} \overset{Eq.1}{\longmapsto} \begin{pmatrix} \frac{-1((t+20)-(-t+20))}{(-t+20)+2+t+20} \\ \frac{-1(-t+20-(t+20))}{(-t+20)+2+t+20} \\ \frac{-2(-1)(0)}{(-t+20)+2+t+20} \end{pmatrix} = \begin{pmatrix} \frac{-2t}{42} \\ \frac{2t}{42} \\ 0 \end{pmatrix} = t \begin{pmatrix} \frac{-2}{42} \\ \frac{2}{42} \\ 0 \end{pmatrix}. \qquad \text{(Eq. 3)}$$

Once again, we see directly from our computation that the image of a line under perspective projection is again a line (because our output consists of multiples of a fixed vector). Now plug parameters $t = -1$ and $t = 1$ into (Eq. 3). We get respectively:

$$\begin{pmatrix} \frac{1}{21} \\ \frac{-1}{21} \\ 0 \end{pmatrix} \text{ and } \begin{pmatrix} \frac{-1}{21} \\ \frac{1}{21} \\ 0 \end{pmatrix}.$$

This further away line segment shrunk even more from length $2\sqrt{2}$ to $(2/21)\sqrt{2}$.

3) Plug the line segment $\left\{ t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 200 \\ 200 \\ 0 \end{pmatrix} : -1 \leq t \leq 1 \right\}$ in:

$$t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 200 \\ 200 \\ 0 \end{pmatrix} = \begin{pmatrix} -t + 200 \\ t + 200 \\ 0 \end{pmatrix} \overset{Eq.1}{\longmapsto} \begin{pmatrix} \frac{-1((t+200)-(-t+200))}{(-t+200)+2+t+200} \\ \frac{-1(-t+200-(t+200))}{(-t+200)+2+t+200} \\ \frac{-2(-1)(0)}{(-t+200)+2+t+200} \end{pmatrix} = \begin{pmatrix} \frac{-2t}{402} \\ \frac{2t}{402} \\ 0 \end{pmatrix} = t \begin{pmatrix} \frac{-2}{402} \\ \frac{2}{402} \\ 0 \end{pmatrix}. \qquad \text{(Eq. 4)}$$

Again plug in the parameter t:

$$\text{for } t = -1, \text{ it gives } \begin{pmatrix} \frac{1}{201} \\ \frac{-1}{201} \\ 0 \end{pmatrix} \text{ and for } t = 1, \text{ it gives } \begin{pmatrix} \frac{-1}{201} \\ \frac{1}{201} \\ 0 \end{pmatrix}.$$

The furthest away line segment shrunk from length $2\sqrt{2}$ to $(2/201)\sqrt{2}$, even smaller. Empirically mapping the points shows that further objects are indeed smaller after applying perspective projection. More generally, the larger the value of $x + y$ (which is what changes when the original object moves parallel to itself in these calculations), the larger the denominator, so the smaller the image object.

3.4 Parallel Lines (along the viewing plane) Stay Parallel & Get Closer

Lines that are parallel along the viewing plane either lie on the plane or never intersect with the plane. $t \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is the line on the viewing plane through the origin. Projected, it would of course remain unchanged because it is already on the viewing plane. But what about projecting other lines parallel to, yet fixed off the viewing plane? Here we argue that projecting such lines does not affect their parallelism, but it still causes them to grow closer. Let's project the following lines, which are all parallel to, yet not on the viewing plane:

$$t\begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}2\\2\\0\end{pmatrix}, \text{ and } t\begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}2\\2\\4\end{pmatrix}.$$

At first, these lines exist 4 units apart because the second line is the first line shifted up four units in the z-direction. But after projection,

$$t\begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}2\\2\\0\end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix}\frac{-t}{3}\\\frac{t}{3}\\0\end{pmatrix};$$

$$t\begin{pmatrix}-1\\1\\0\end{pmatrix} + \begin{pmatrix}2\\2\\4\end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix}\frac{-t}{3}\\\frac{t}{3}\\\frac{4}{3}\end{pmatrix}.$$

They are $\frac{4}{3}$ units apart; they grew closer!

3.5   Parallel Lines (not along the viewing plane) Meet

The line $t\begin{pmatrix}1\\1\\0\end{pmatrix}$, through the origin, is perpendicular to the viewing plane $x + y = 0$. Here we argue that *any* line perpendicular to the plane of projection through the point $(x, -x, z)$ appears to meet at the point $(0, 0, 0)$ upon projection, as we project the points going to infinity. Let's project the following three lines, which are all perpendicular to the viewing plane:

$$t\begin{pmatrix}1\\1\\0\end{pmatrix}, t\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}-1\\1\\0\end{pmatrix}, \text{ and } t\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

It should be apparent that every point along the first line, $t\begin{pmatrix}1\\1\\0\end{pmatrix}$ collapses to $(0, 0, 0)$, since $t\begin{pmatrix}1\\1\\0\end{pmatrix}$ is the line that is perpendicular to the viewing plane and also connects to the center of projection. That is, when we follow the projection recipe, creating a line between the center of projection and each input point on $t\begin{pmatrix}1\\1\\0\end{pmatrix}$, and sliding along that line until to its intersection with the viewing plane, each input point maps to $(0, 0, 0)$. Even more interestingly though, is that even the perpendicular lines fixed off of the origin appear to approach or meet at the point $(0, 0, 0)$ as one moves to infinity along the line — that is, the point $(0, 0, 0)$ is a limit. If we draw lines from the center of projection to the point going to infinity on the perpendicular lines, the lines encroach on the viewing plane at the point $(0, 0, 0)$. We will plug the lines into the perspective projection function (with our center of projection still at $(-1, -1, 0)$) to demonstrate this effect.

$$t\begin{pmatrix}1\\1\\0\end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix}0\\0\\0\end{pmatrix};$$

$$t\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}-1\\1\\0\end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix}\frac{-2}{2t+2}\\\frac{2}{2t+2}\\0\end{pmatrix} = \frac{1}{t+1}\begin{pmatrix}-1\\1\\0\end{pmatrix};$$

$$t\begin{pmatrix}1\\1\\0\end{pmatrix} + \begin{pmatrix}1\\-1\\0\end{pmatrix} \xmapsto{Eq.1} \begin{pmatrix}\frac{2}{2t+2}\\\frac{-2}{2t+2}\\0\end{pmatrix} = \frac{1}{t+1}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Note that the image of the input line under perspective projection is once again a line but missing one point, since we see from the calculation that the output vectors are all all *nonzero* multiples of a fixed vector as we plug in various values for t. The missing point can be accounted for. As we take the point we are projecting to infinity along these different lines (i.e., taking the limit as $t \to \infty$), we get:

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$$\lim_{t\to\infty}\begin{pmatrix}0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix};$$

$$\lim_{t\to\infty}\begin{pmatrix}\frac{-2}{2t+2}\\\frac{2}{2t+2}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix};$$

$$\lim_{t\to\infty}\begin{pmatrix}\frac{2}{2t+2}\\\frac{-2}{2t+2}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}.$$

Hence, moving along these parallel lines (off the viewing plane) to infinity is like getting closer and closer to a meeting point.

The drawing below, in two point perspective, illustrates this effect. It shows that vanishing points are not limited to parallel lines perpendicular to the viewing plane. Actually there exists vanishing points for every direction not parallel to the viewing plane. The drawing below illustrates two vanishing points for two different sets of parallel lines — hence the name.
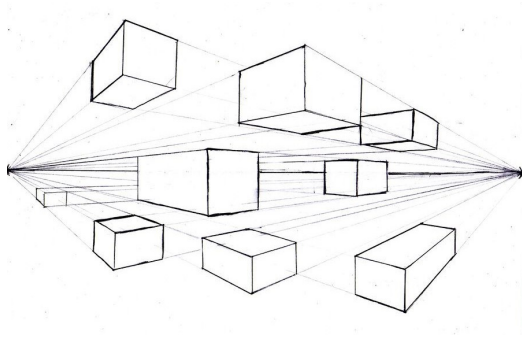

Figure 4: *Two Point Perspective*: parallel lines not along the viewing plane meet at 'vanishing points'.

3.6   Moving the Center of Projection to Infinity

To show the effect of moving the center of projection to infinity, we first rearrange out projection formula:

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\xmapsto{Eq.1}\begin{pmatrix}\frac{d(y-x)}{x-2d+y}\\\frac{d(x-y)}{x-2d+y}\\\frac{-2dz}{(x-2d+y)}\end{pmatrix}=\begin{pmatrix}\frac{(y-x)}{\frac{x+y}{d}-2}\\\frac{(x-y)}{\frac{x+y}{d}-2}\\\frac{-2z}{\frac{x+y}{d}-2}\end{pmatrix}.$$

Taking the limit as d goes to infinity gives rise to the the following transformation,

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\xmapsto[d\to\infty]{Eq.1}\begin{pmatrix}\frac{x-y}{2}\\\frac{y-x}{2}\\z\end{pmatrix}.$$

The output of the transformation lies on the viewing plane $x+y=0$ because the first two coordinates of the output vector add to 0. This linear transformation above actually represents the orthogonal projection onto the $x+y=0$ plane in $\mathbb{R}^3$! The difference between the input vector and the output vector,

$$\begin{pmatrix}x\\y\\z\end{pmatrix}-\begin{pmatrix}\frac{x-y}{2}\\\frac{y-x}{2}\\z\end{pmatrix}=\begin{pmatrix}\frac{2x-(x-y)}{2}\\\frac{2y-(y-x)}{2}\\z-z\end{pmatrix}=\begin{pmatrix}\frac{x+y}{2}\\\frac{x+y}{2}\\0\end{pmatrix}=\left(\frac{x+y}{2}\right)\begin{pmatrix}1\\1\\0\end{pmatrix},$$

a multiple of $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, lies on the line perpendicular to the plane $x + y = 0$, through the origin. We see that sending the center of projection to infinity renders the perspective obsolete, turning the *perspective* projection into an *orthogonal* one.

4   Conclusions & Applications

In life we experience things from our own perspective. The math of perspective projections appreciates and accounts for this fact. Perspective Projections are ubiquitous in real life application. They appear in art, graphics, and biology.

In this paper, we worked through *the perspective projection that moves points in $\mathbb{R}^3$ to the plane $x + y = 0$, from the point $(d, d, 0)$*. It should be noted that this is one particular projection. We could also, for example, find the perspective projection that moves points in $\mathbb{R}^3$ to the plane $z = 0$, from any point (x, y, z). Or we could find the perspective projection that moves points in $\mathbb{R}^4$ onto any 2D plane, from a center of projection that is a line rather than a point. Generally, the dimension of the viewing plane and the dimension of the center of projection add up to one less than the ambient dimension. Some key points to heed from this paper are that: perspective projections take into account a perspective from which the projection occurs, and unlike orthogonal projections, perspective projections are non-linear and do not map parallel lines to parallel lines.

Moving forward, I am curious about applying these projections to the realm of computer graphics, computer vision, and mathematical biology.

5 References

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